

Proving geometric inequalities

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Abstract

We introduce *GeoGebra Discovery* that can automatically prove or discover geometric inequalities. It consists of

- an extended version of *GeoGebra*,
- a controller web service *realgeom*,
- and the computational tools *Mathematica* or *Tarski* (that embeds *QEPCAD B*).

We successfully solve several non-trivial problems in Euclidean planar geometry via a simple graphical user interface.

Some parts of this talk are joint work with
Róbert Vajda, Tomás Recio, M. Pilar Vélez, Simone Luksch,
Celina Abar, Alexandre M. Russo and other colleagues.

Related and preliminary work

- J. Robu, “Automated proof of geometry theorems involving order relation in the frame of the *Theorema* project,” in *Proceedings of the International Conference on Knowledge Engineering, Principles and Techniques, KEPT2007, Cluj-Napoca (Romania), June 6–8, 2007*, 2007, pp. 307–315
- C. A. A. P. Abar, Z. Kovács, T. Recio, and R. Vajda, *Conectando Mathematica e GeoGebra para explorar construções geométricas planas*, Presentation at Wolfram Technology Conference, São Paulo, Brazil, Nov. 23, 2019. [Online]. Available: https://www.researchgate.net/publication/337499551_Conectando_Mathematica_e_GeoGebra_para_explorar_construcoes_geometricas_planas
- R. Vajda and Z. Kovács, “GeoGebra and the *realgeom* reasoning tool,” in *PAAR+SC-Square 2020. Workshop on Practical Aspects of Automated Reasoning and Satisfiability Checking and Symbolic Computation Workshop 2020*, P. Fontaine, K. Korovin, I. S. Kotsireas, P. Rümmer, and S. Tourret, Eds., 2752 vols., Nov. 28, 2020, pp. 204–219. eprint: <http://ceur-ws.org/Vol-2752/paper15.pdf>. [Online]. Available: <https://doi.org/urn:nbn:de:0074-2752-0>
- J. Heimrath, *The area method in the Wolfram language. Presentation at ADG 2021*,
https://www.youtube.com/watch?v=X28_HYCzpoM&t=2510s, 2021

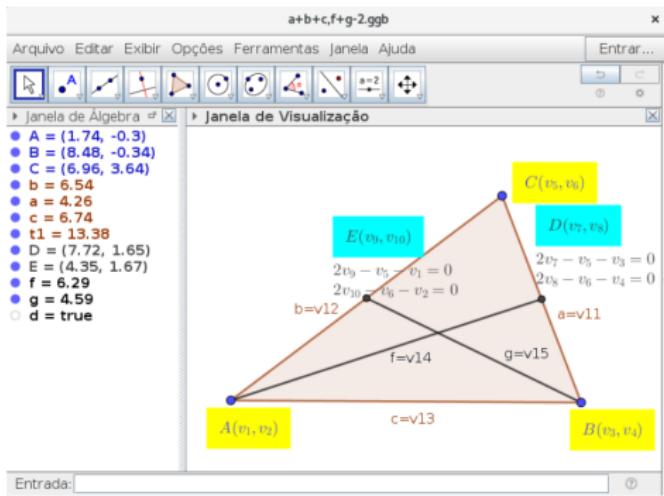
Preliminary work (Abar, Kovács, Recio, and Vajda, 2019)

Conectando Mathematica e GeoGebra para explorar construções geométricas planas



Preliminary work (Abar, Kovács, Recio, and Vajda, 2019)

Conectando Mathematica e GeoGebra para explorar construções geométricas planas



- Todos esses dados, e também os lados esquerdo e direito da questão da comparação ($w1=v11+v12+v13$, $w2=v14+v15$) são enviados para o realgeom.
- A ferramenta realgeom constrói um comando Mathematica. Acrescenta a igualdade $w1==w2$ e algumas desigualdades que descrevem a situação geométrica em detalhe, porque estamos interessados no sentido matemático nesta questão: Para que números m pode a equação $w1sm^w2$ ser resolvida?
- Após estes passos realgeom obtém a solução do sistema de igualdade e desigualdade através da comunicação da linha de comando com Mathematica:

```
Print[Quiet[Reduce[Resolve[Exists[{  
v1,v2,v3,v4,v5,v6,v7,v8,v9,v10,v11,v12,v13,v14,v15,w1,w2],  
(w>0) \[And] (w1 == m + w2)]  
\[And] (v14>0) \[And] (v15>0) \[And] (v13>0) \[And] (v12>0)  
\[And] (2xv7-v5-1==0) \[And] (2xv8-v6==0) \[And] (2xv9-v5==0)  
\[And] (-v13^2+2xv6^2+xv5^2-2xv5+1==0) \[And] (-v14^2+2xv10^2+xv8^2-2xv9+1==0)  
\[And] (-v11^2+2xv6^2+xv5^2==0) \[And] (-v12^2+2xv11^2==0) \[And] (-v13^2+2xv8^2+xv7^2==0)  
\[And] (-w1+v13+v11+v12==0) \[And] (-w2+v15+v14==0)],Reals]]\[InputForm]]
```

Preliminary work (Abar, Kovács, Recio, and Vajda, 2019)

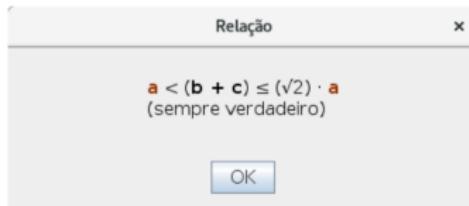
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```
In[1]:= f[m] = Print[Quiet@Reduce@Resolve@Exists[{v16, v17, v18, v19, v20, v21, v23, v24, v25, v26, w2},  
m > 0 && v24 == m w2 && v20 > 0 && v24 > 0 && v26 > 0 &&  
v23^2 - v24^2 + 1 == 0 && v23^2 - v25^2 == 0 && -v26^2 + 1 == 0  
&& v25 + v26 - w2 == 0], Reals] // InputForm]
```

During evaluation of In[1]:=

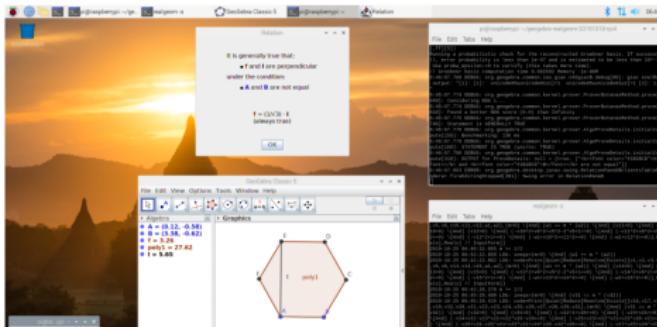
```
Inequality[1/Sqrt[2], LessEqual, m, Less, 1]
```

- Ao alterar a ordem das entradas, a saída é ainda mais legível:



- A primeira parte da desigualdade é uma consequência imediata da desigualdade do triângulo.

Demonstração em um Raspberry Pi 3



Related work (Heimrath, 2021)

The Area Method in the Wolfram Language. Presentation at ADG 2021

The Triangle Inequality

```
(*) Constructing a triangle . *)
ClearAll[triangle];
triangle = FreePoint[a, b, c]
```

Out[1] = \triangle

```
(*) Stating the conjecture . *)
ClearAll[triangleInequality];
triangleInequality = (Sqrt[PythagoreanDifference[a, b, a]/2] + Sqrt[PythagoreanDifference[b, c, b]/2]) >= Sqrt[PythagoreanDifference[a, c, a]/2]
```

Out[2] = $\frac{\sqrt{\mathcal{P}_{a\,b\,a}}}{\sqrt{2}} + \frac{\sqrt{\mathcal{P}_{b\,c\,b}}}{\sqrt{2}} \geq \frac{\sqrt{\mathcal{P}_{a\,c\,a}}}{\sqrt{2}}$

```
(*) Proving the inequality . *)
VerifyConjecture[triangleInequality, triangle, "UseAreaCoordinates" -> True, "ProofNotebook" -> True]
```

Conjecture :

$$\frac{\sqrt{\mathcal{P}_{a\,b\,a}}}{\sqrt{2}} + \frac{\sqrt{\mathcal{P}_{b\,c\,b}}}{\sqrt{2}} \geq \frac{\sqrt{\mathcal{P}_{a\,c\,a}}}{\sqrt{2}}$$

Eliminated Point	Applicable Substitutions	Current Conjecture Statement
-	Expanding the definition of \mathcal{P} . Transition to Area Coordinates: $\overline{ab}^2 \rightarrow 1$	$\sqrt{\overline{ab}^2} + \sqrt{\overline{bc}^2} \geq \sqrt{\overline{ac}^2}$
-	$\overline{bc}^2 \rightarrow \frac{\overline{ab}^2 \overline{a\,c\,c}^2 + \overline{ac}^2 \overline{a\,b\,b}^2}{\overline{a\,b\,c}^2}$	True
	$\overline{ac}^2 \rightarrow 1$	

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GeoGebra Discovery: an experimental version of GeoGebra

github.com/kovzol/geogebra-discovery

Feature	GeoGebra	GeoGebra Discovery	Next step
Discover tool/command	no	yes	● Scheduled for merging into GeoGebra
Compare command	no	yes	● GeoGebra Team: approve/update
IncircleCenter command	no	yes (with prover support)	● GeoGebra Team: approve (discuss Center(Incircle) first)
Incircle tool	no	yes	● GeoGebra Team: approve/update
IncircleCenter tool	no	yes	● GeoGebra Team: approve/update
LocusEquation tool	no	yes	● GeoGebra Team: approve/update
Envelope tool	no	yes	● GeoGebra Team: approve/update
Raspberry Pi 3D View	no	yes	● GeoGebra Team: approve/update
Java OpenGL	2.2	2.4	● GeoGebra Team: approve/update
Giac: threads on Linux	no	yes	● GeoGebra Team: approve/update
Same color for circles with the same radius	no	yes	● GeoGebra Team: approve/update
Proving inequalities	no	yes	● Use Tarski as a dynamic library
ApplyMap command	no	prototype	● Fix bugs and make improvements

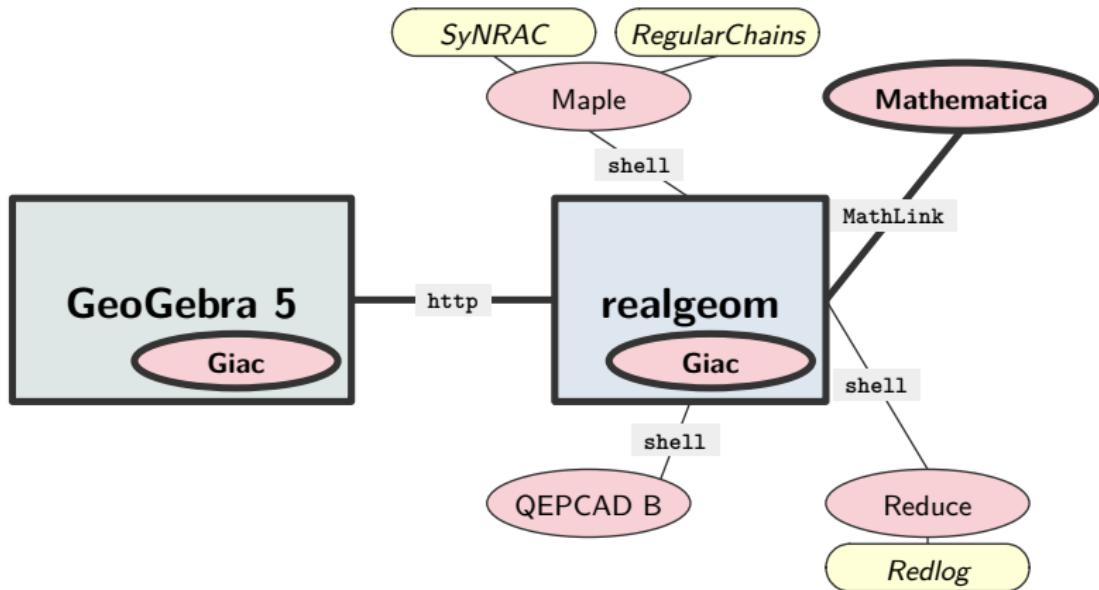
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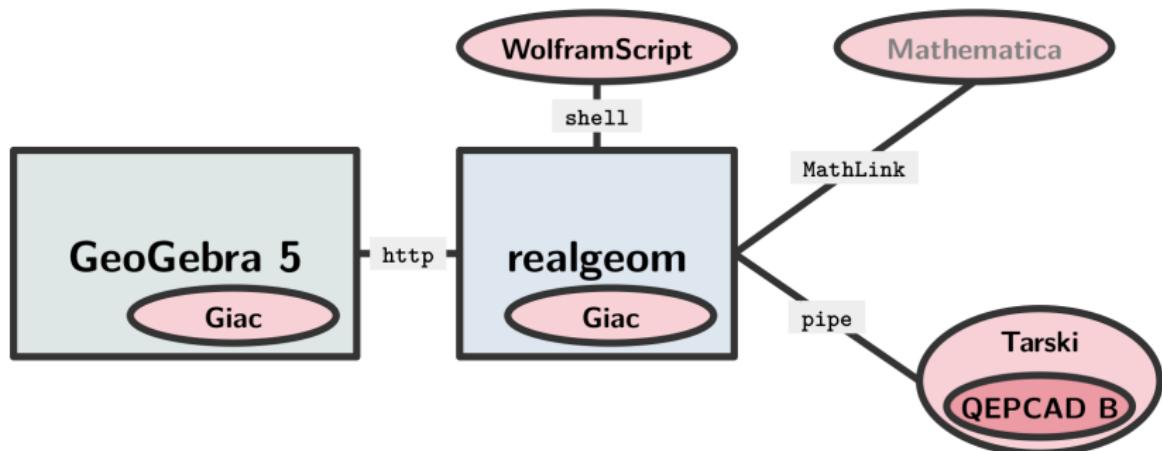
Implementation: System layout of GeoGebra Discovery

September 2020, Vajda and Kovács, "GeoGebra and the *realgeom* Reasoning Tool"



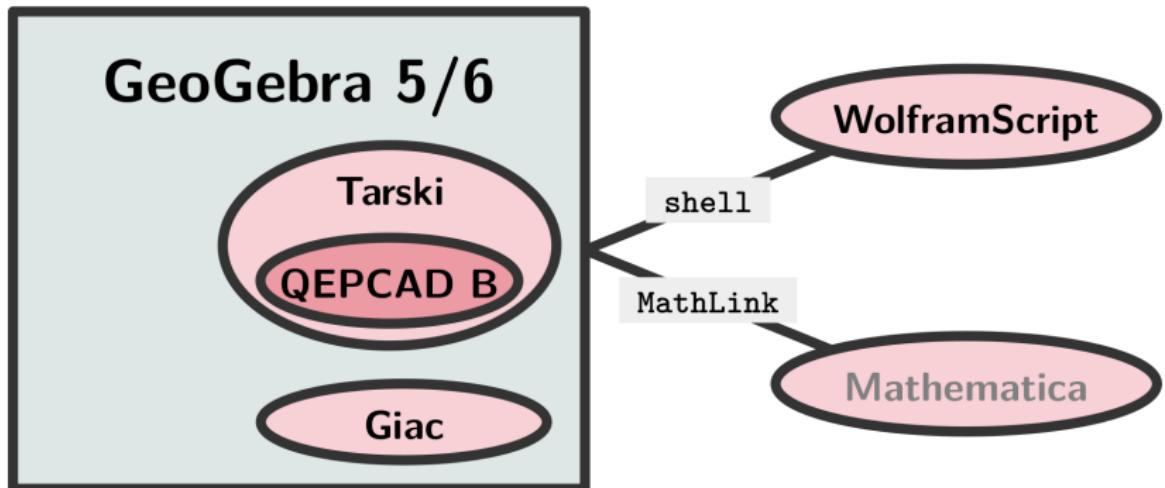
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July 2021



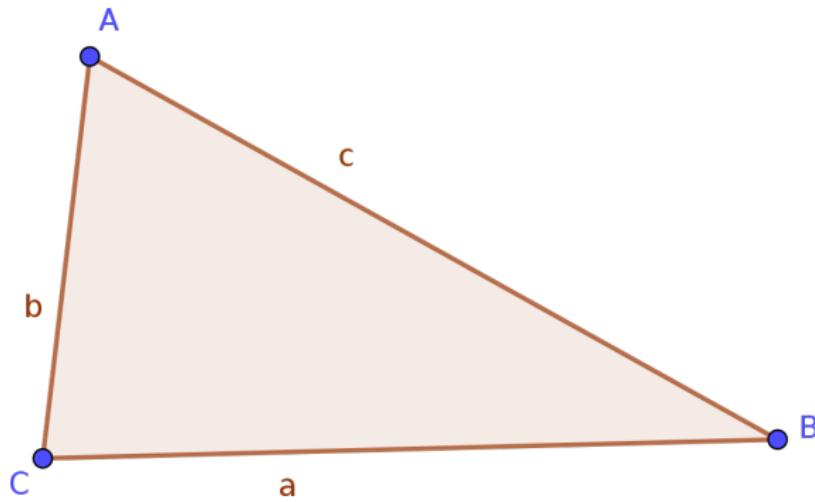
Implementation: System layout of GeoGebra Discovery

Planned, on-going work



Motivation

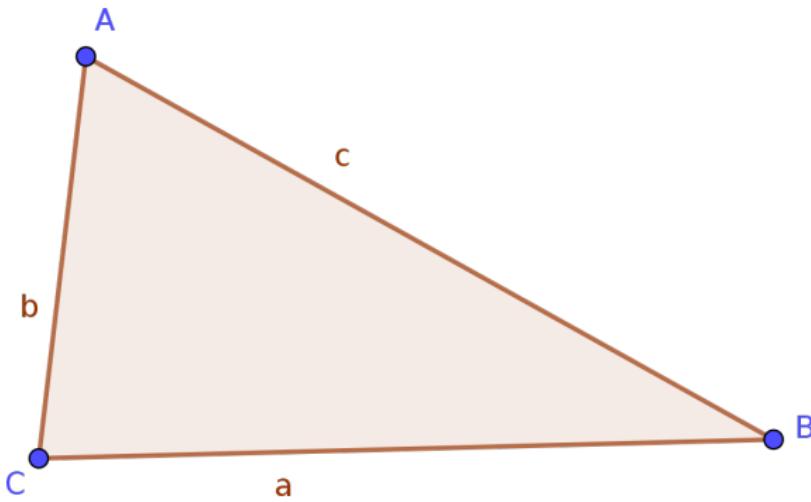
The Pythagorean Theorem... a generalization



$$a^2 + b^2 \dots$$

Motivation

The Pythagorean Theorem... a generalization

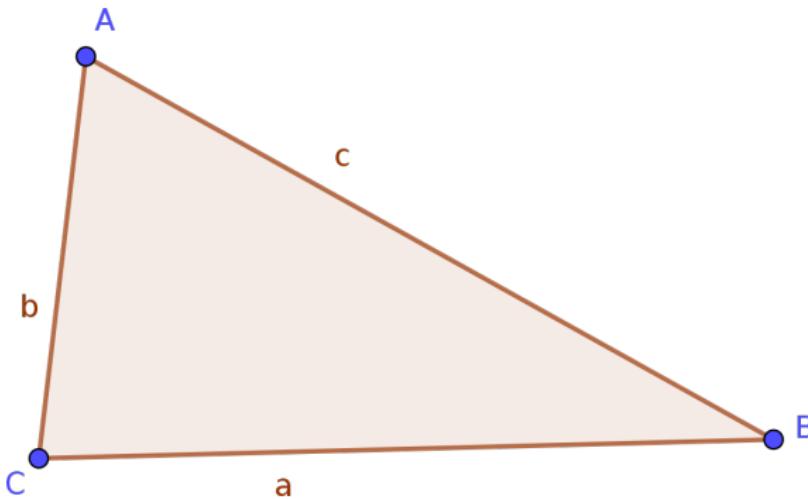


$$a^2 + b^2 \dots$$

$$= c^2?$$

Motivation

The Pythagorean Theorem... a generalization



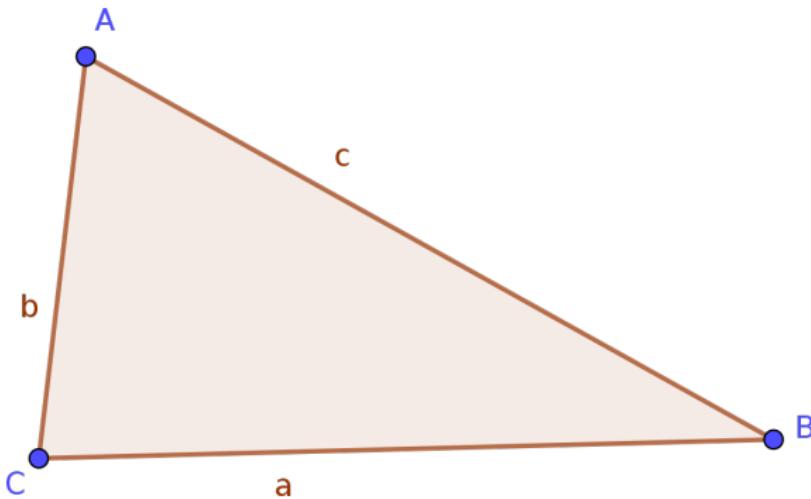
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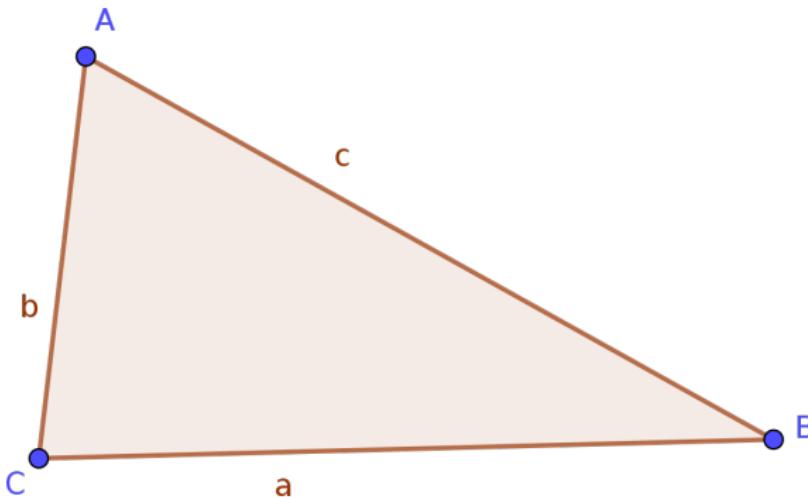
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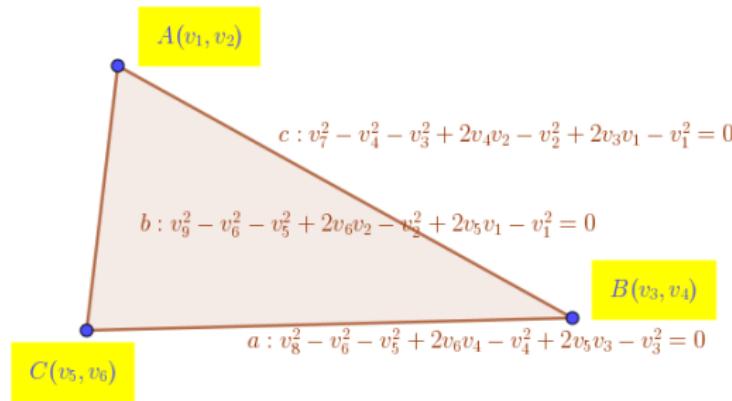
$$< c^2?$$

something
else?

Motivation

A generalization of the Pythagorean Theorem

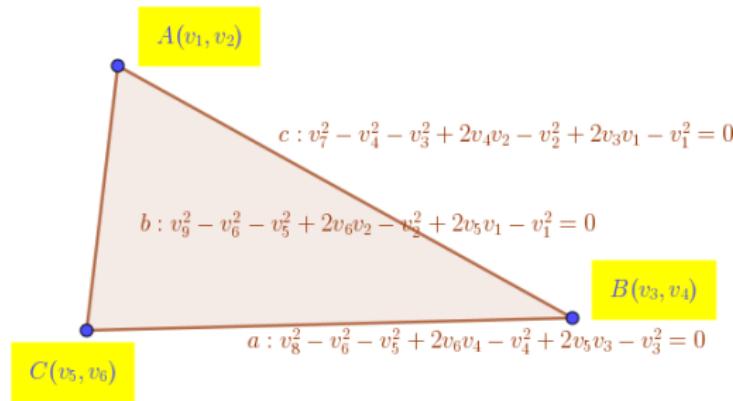
① Equational hypotheses:



Motivation

A generalization of the Pythagorean Theorem

① Equational hypotheses:



② Non-degeneracy condition:

$$v_{10} \cdot (v_5 \cdot v_4 - v_6 \cdot v_3 - v_5 \cdot v_2 + v_3 \cdot v_2 + v_6 \cdot v_1 - v_4 \cdot v_1) = 1$$

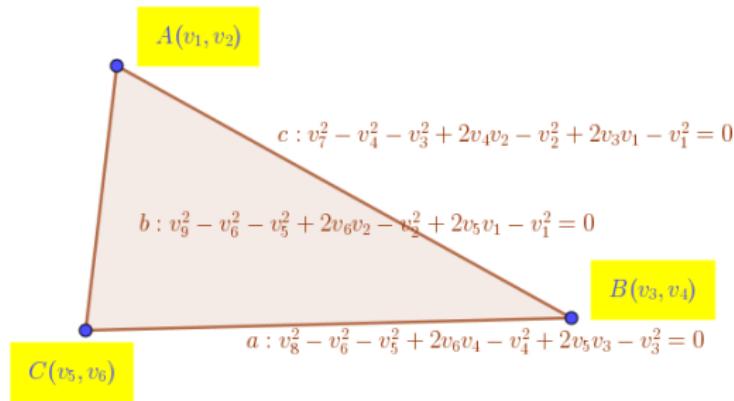
③ Exploration related equation: $\mu \cdot v_7^2 = v_8^2 + v_9^2$

④ Non-equational assumptions: $v_7 > 0 \wedge v_8 > 0 \wedge v_9 > 0$

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A generalization of the Pythagorean Theorem

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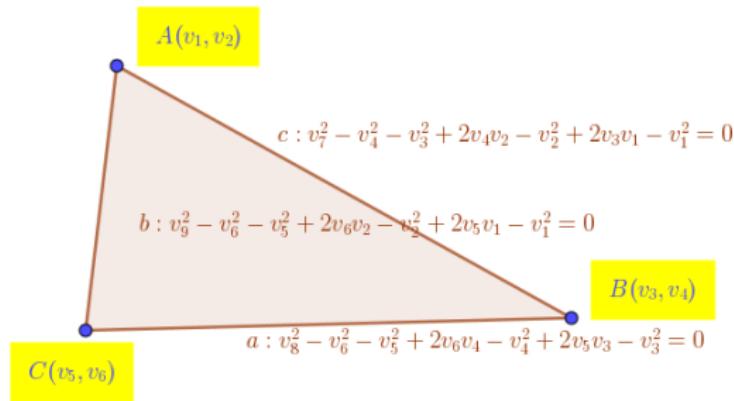
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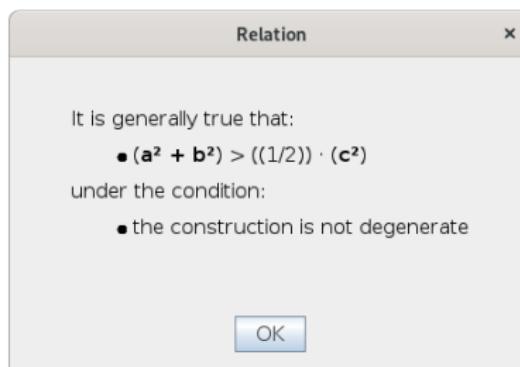
④ Non-equational assumptions: $v_7 > 0 \wedge v_8 > 0 \wedge v_9 > 0$

$$\Rightarrow \mu > 1/2$$

Motivation

A generalization of the Pythagorean Theorem

Symbolic check in GeoGebra (via `Relation(a2 + b2, c2)`):



- ① Exploration related equation:

$$Q_1 = \mu \cdot Q_2$$

where Q_1 and Q_2 are the geometric quantities to compare and $\mu \in \mathbb{R}$ is a new variable (“proportion” or “ratio”).

- ② Derivation of an equivalent form of the (semi-)algebraic system:

- ① elimination via Gröbner bases, for algebraic systems,
- ② cylindrical algebraic decomposition (CAD) and real quantifier elimination (RQE), for semi-algebraic systems.

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$$\Rightarrow m \cdot Q_2 \leq Q_1 \leq M \cdot Q_2$$

(=) (=)

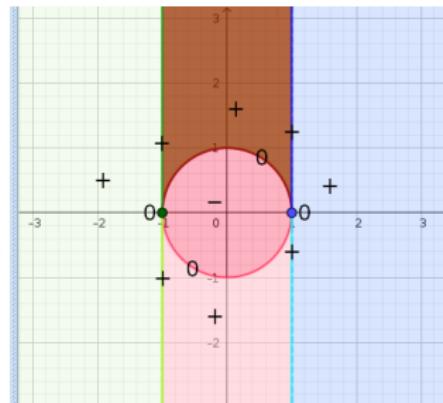
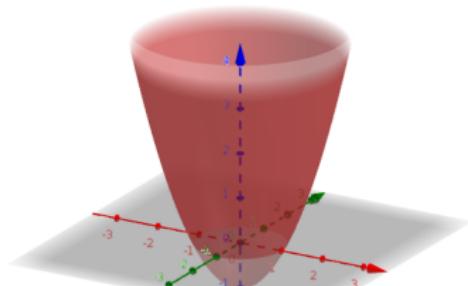
where $m, M \in \mathbb{R}_0^+$ are sharp constants.

A semi-algebraic technique

Cylindrical Algebraic Decomposition (CAD) and Real Quantifier Elimination (RQE)

Definition

Given a set S of polynomials in $\mathbb{Z}[x_1, x_2, \dots, x_n]$, a CAD is a decomposition of \mathbb{R}^n into special connected semi-algebraic sets, on which each polynomial has constant sign, either $+$, $-$ or 0.



Example: $S = \{x_1^2 + x_2^2 - 1\}$ and a CAD of it. Here \mathbb{R}^2 can be decomposed into 13 semi-algebraic sets ($13 = 1 + 3 + 5 + 3 + 1$).

- G. Collins, “Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition,” in *Lecture Notes in Computer Science*, vol. 33, 1975, pp. 134–183
- A. Strzeboński, “Cylindrical algebraic decomposition using local projections,” *Journal of Symbolic Computation*, vol. 76, pp. 36–64, Sep. 2016
- F. Vale-Enriquez and C. Brown, “Polynomial constraints and unsat cores in TARSKI,” in *Mathematical Software – ICMS 2018. LNCS*, vol. 10931, Springer, Cham, 2018, pp. 466–474

Reformulating the problem as input for RQE (via CAD)

Generalization of the Pythagorean theorem, 280/1714 ms (MathLink/shell*)

The quantified formula (after simplifying):

$$\begin{array}{l} \exists \mu > 0 \wedge v_7 > 0 \wedge v_8 > 0 \wedge v_9 > 0 \wedge \\ v_{10}, v_5, v_6, v_7, v_8, v_9 \in \mathbb{R} \\ v_{10}v_6 - 1 = 0 \wedge -v_5^2 - v_6^2 + v_7^2 = 0 \wedge \\ -v_5^2 + 2v_5 - v_6^2 + v_8^2 - 1 = 0 \wedge \\ -v_9 + 1 = 0 \wedge -\mu + v_7^2 + v_8^2 = 0. \end{array}$$

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```
In[1]:= ToRadicals[
  Reduce[
    Resolve[Exists[{v10, v5, v6, v7, v8, v9},
      (m > 0) \wedge (v7 > 0) \wedge (v8 > 0) \wedge (v9 > 0) \wedge (v10 * v6 - 1 == 0) \wedge (-v5^2 - v6^2 + v7^2 == 0) \wedge
      (-v5^2 + 2 * v5 - v6^2 + v8^2 - 1 == 0) \wedge (-v9 + 1 == 0) \wedge (-m + v7^2 + v8^2 == 0)], Reals],
    Reals], Cubics \rightarrow False]
```

```
Out[1]= m > 1/2
```

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    Reals], Cubics \rightarrow False]
Out[1]= m > 1/2
```

$\Rightarrow \mu > 1/2$ (a quantifier-free formula).

Additional ways for users to enter input

. . . instead of using Relation($a^2 + b^2, c^2$)

- ① Direct proof by typing `Prove($a^2 + b^2 > c^2/2$)`, or by trial-and-error:
 - e.g. `Prove($a^2 + b^2 > c^2$)`

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 - ...

Additional ways for users to enter input

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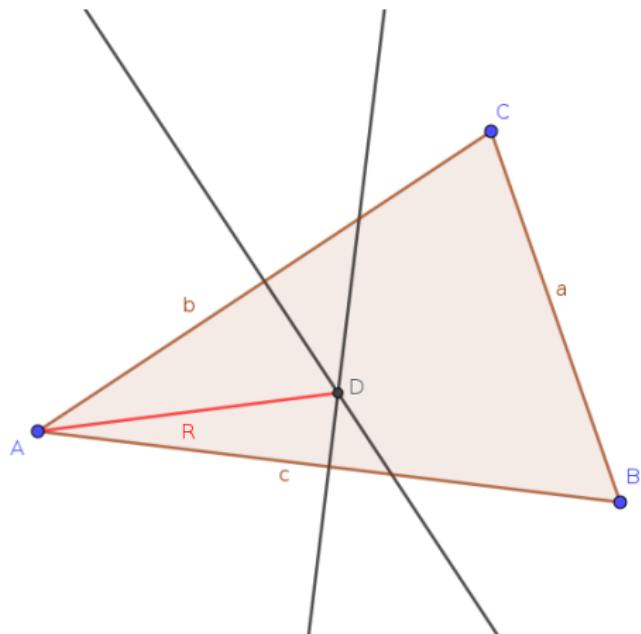
- ➊ Direct proof by typing Prove($a^2 + b^2 > c^2/2$), or by trial-and-error:
 - e.g. Prove($a^2 + b^2 > c^2$) → **false**,
 - e.g. Prove($a^2 + b^2 > c^2/3$) → **true**,
 - ...
- ➋ Low-level command Compare($a^2 + b^2, c^2$) to get direct result
(→ JavaScript API)

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- ➌ In simpler cases: point-and-click (via the Relation tool)

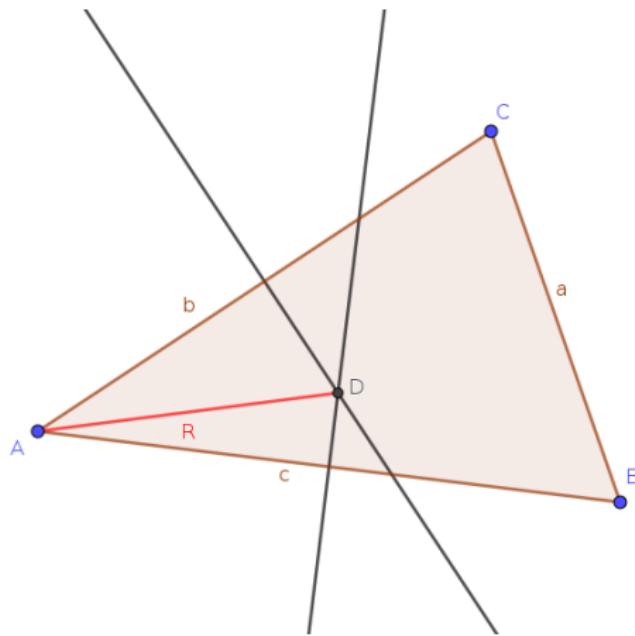
Bottema's problem 5.3



If the lengths of the sides of a triangle are a , b , c , and the length of its circumradius is R , then

$$a + b + c$$

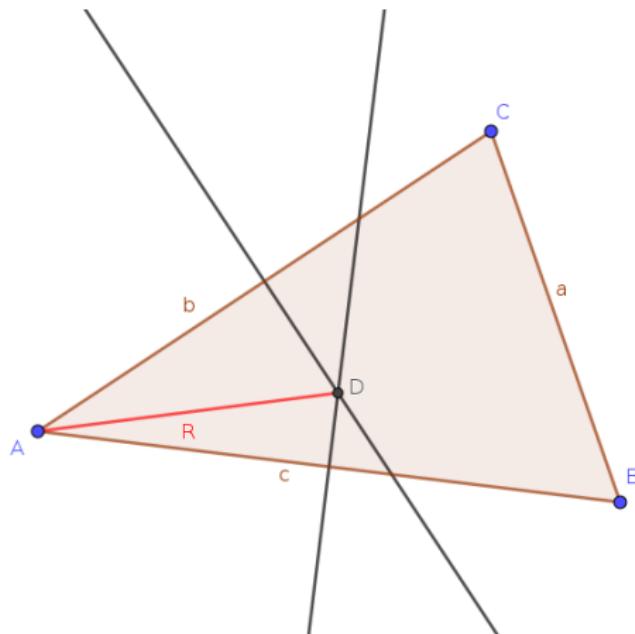
Bottema's problem 5.3



If the lengths of the sides of a triangle are a , b , c , and the length of its circumradius is R , then

$$a + b + c \leq$$

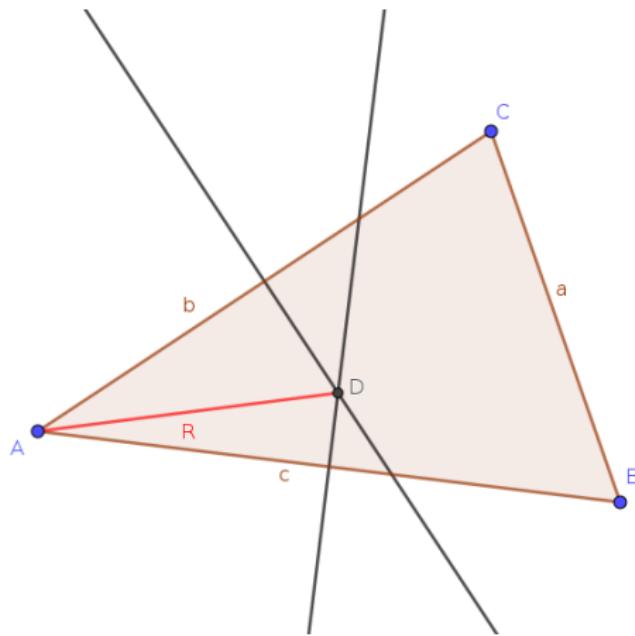
Bottema's problem 5.3



If the lengths of the sides of a triangle are a , b , c , and the length of its circumradius is R , then

$$a + b + c \leq 3R$$

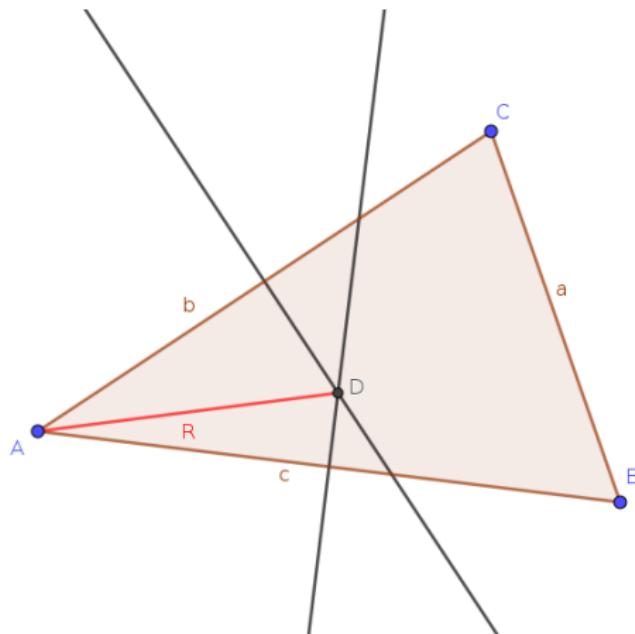
Bottema's problem 5.3



If the lengths of the sides of a triangle are a , b , c , and the length of its circumradius is R , then

$$a + b + c \leq 3\sqrt{3}$$

Bottema's problem 5.3



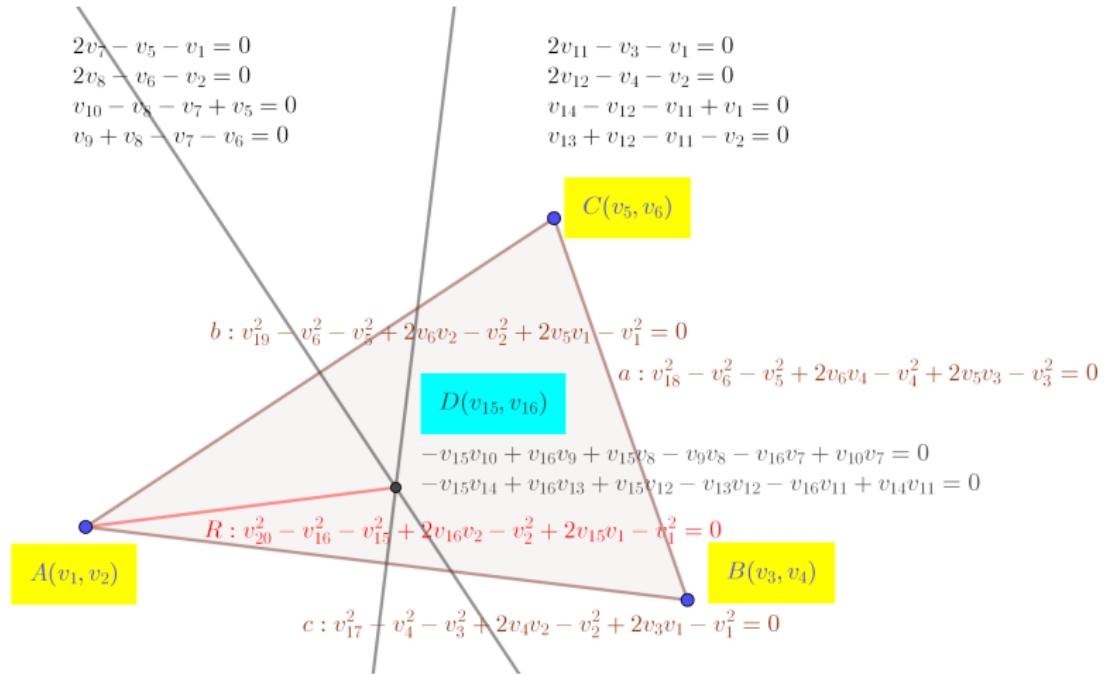
If the lengths of the sides of a triangle are a , b , c , and the length of its circumradius is R , then

$$a + b + c \leq 3\sqrt{3} R.$$

Bottema's problem 5.3

Comparison of the perimeter and the circumradius in a triangle

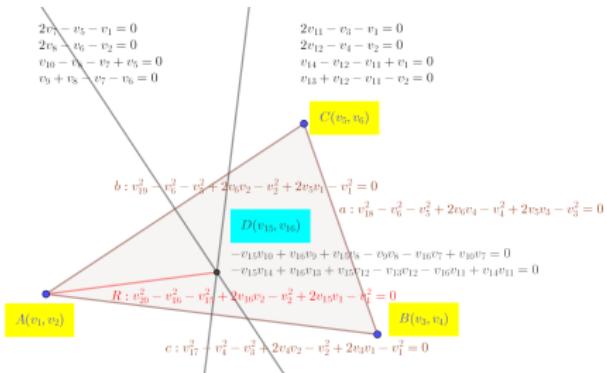
① Equational hypotheses:



Bottema's problem 5.3

Comparison of the perimeter and the circumradius in a triangle

1 Equational hypotheses:



2 Non-degeneracy condition:

$$v_{21} \cdot (v_5 \cdot v_4 - v_6 \cdot v_3 - v_5 \cdot v_2 + v_3 \cdot v_2 + v_6 \cdot v_1 - v_4 \cdot v_1) = 1$$

3 Exploration related equation: $\mu \cdot v_{20} = v_{17} + v_{18} + v_{19}$

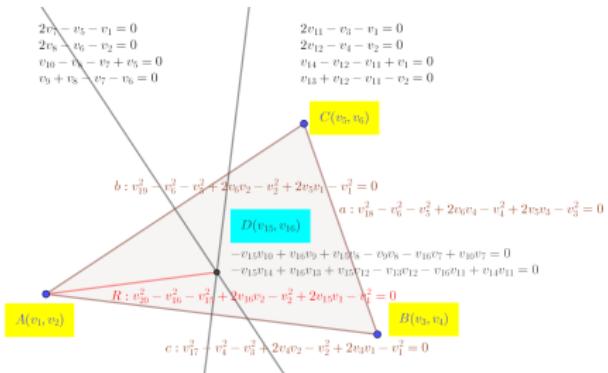
4 Non-equational assumptions:

$$v_{17} > 0 \wedge v_{18} > 0 \wedge v_{19} > 0 \wedge v_{20} > 0$$

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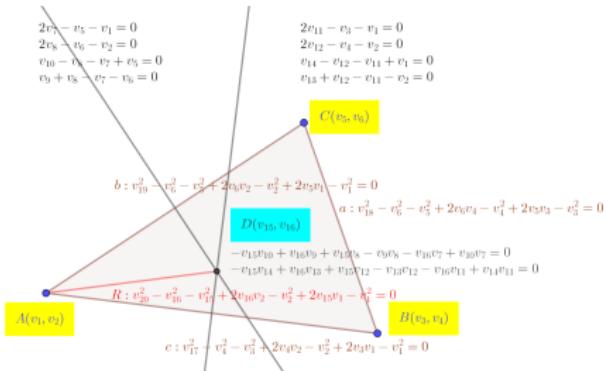
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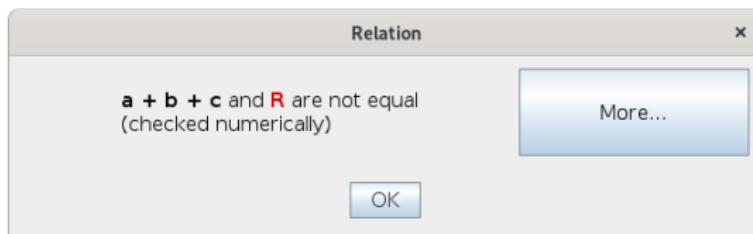
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$$\Rightarrow \mu \leq 3\sqrt{3}$$

Bottema's problem 5.3

Comparison of the perimeter and the circumradius

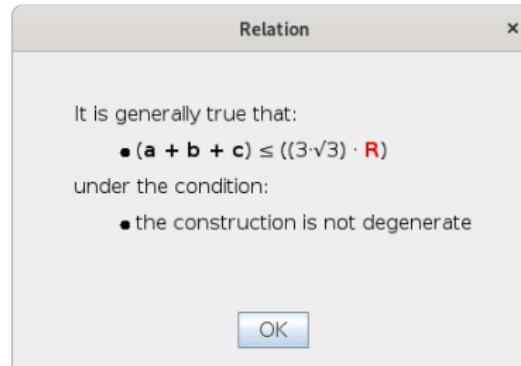
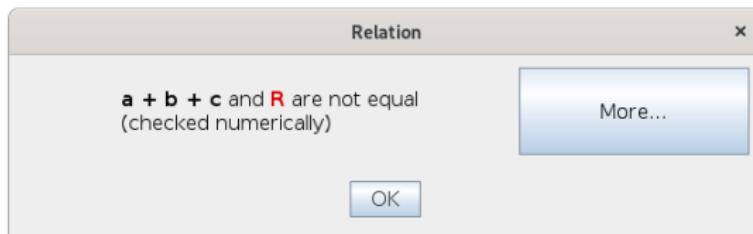
Symbolic check in GeoGebra (via the Relation tool):



Bottema's problem 5.3

Comparison of the perimeter and the circumradius

Symbolic check in GeoGebra (via the Relation tool):



Reformulating the problem as input for RQE

Bottema 5.3, 3529/2532 ms (MathLink/shell*)

The quantified formula (after simplifying):

$$\begin{array}{l} \exists \mu > 0 \wedge v_{17} > 0 \wedge v_{18} > 0 \wedge v_{19} > 0 \wedge \\ v_{10}, v_{16}, \dots, v_{21}, v_5, v_6, v_9, w_1 \in \mathbb{R} \\ v_{20} > 0 \wedge 2v_{10}v_5 - 2v_{10} - 2v_5v_{16} + 4v_{16}v_9 - 2v_9v_6 + v_6 = 0 \wedge \\ -v_5 - v_6 + 2v_9 = 0 \wedge 2v_{10} + v_5 - v_6 = 0 \wedge v_{21}v_6 - 1 = 0 \wedge \\ v_{18} + v_{20} - w_1 + 1 = 0 \wedge -4v_{16}^2 + 4v_{17}^2 - 1 = 0 \wedge \\ v_{18}^2 - v_5^2 + 2v_5 - v_6^2 - 1 = 0 \wedge v_{20}^2 - v_5^2 - v_6^2 = 0 \wedge \\ -\mu \cdot v_{17} + w_1 = 0 \wedge -v_{19} + 1 = 0 \end{array}$$

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```
In[1]:= ToRadicals[
  Reduce[Resolve[Exists[{v10, v16, v17, v18, v19, v20, v21, v5, v6, v9, w1},
    (m > 0) \wedge (v17 > 0) \wedge (v18 > 0) \wedge (v19 > 0) \wedge (v20 > 0) \wedge
    (2 * v10 * v5 - 2 * v10 - 2 * v5 * v16 + 4 * v16 * v9 - 2 * v9 * v6 + v6 == 0) \wedge (-v5 - v6 + 2 * v9 == 0) \wedge
    (2 * v10 + v5 - v6 == 0) \wedge (v21 * v6 - 1 == 0) \wedge (v18 + v20 - w1 + 1 == 0) \wedge (-4 * v16^2 + 4 * v17^2 - 1 == 0) \wedge
    (v18^2 - v5^2 + 2 * v5 - v6^2 - 1 == 0) \wedge (v20^2 - v5^2 - v6^2 == 0) \wedge (-m * v17 + w1 == 0) \wedge
    (-v19 + 1 == 0)], Reals], Reals], Cubics -> False]

Out[1]= 0 < m \leq 3 \sqrt{3}
```

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    (-v19 + 1 == 0)], Reals], Reals], Cubics -> False]
```

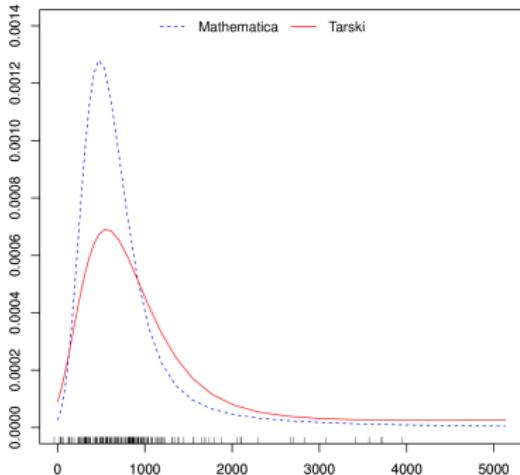
Out[1]= $0 < m \leq 3\sqrt{3}$

$$\Rightarrow \mu \leq 3\sqrt{3} \text{ (a quantifier-free formula).}$$

- 131 simple/moderate tests Database
 - 117/115 can be successfully solved (Mathematica/Tarski) within 30 seconds

Benchmarks

- 131 simple/moderate tests Database
 - 117/115 can be successfully solved (Mathematica/Tarski) within 30 seconds

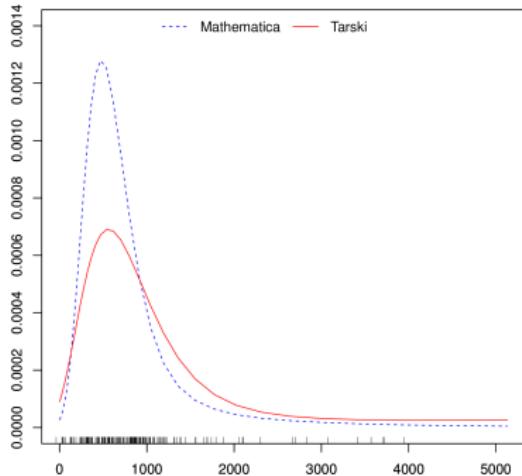


Density estimate on 103 tests that work uniformly (timing in ms),

$$\mu_M = 1361, \mu_T = 2841, \sigma_M = 3379, \sigma_T = 4616$$

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Density estimate on 103 tests that work uniformly (timing in ms),

$$\mu_M = 1361, \mu_T = 2841, \sigma_M = 3379, \sigma_T = 4616$$

- 46 additional tests to prove a given conjecture Database
 - 33/35 can be successfully proven (Mathematica/Tarski) within 60 seconds

Speed issues: how do you draw your construction?

It does matter which bisectors and radius are chosen!

Bottema 5.3

Mathematica

Linux amd64	b, c	a, c	a, b
$R = AD$	2532	2651	t/o
$R = BD$	t/o	2533	t/o
$R = CD$	t/o	2554	t/o

Raspbian 10	b, c	a, c	a, b
$R = AD$	3529	3675	t/o
$R = BD$	t/o	3660	t/o
$R = CD$	t/o	3706	t/o

Tarski

Linux amd64	b, c	a, c	a, b
$R = AD$	631	t/o	1351
$R = BD$	t/o	t/o	1382
$R = CD$	t/o	t/o	t/o

Chrome WASM	b, c	a, c	a, b
$R = AD$	6886	t/o	16717
$R = BD$	t/o	t/o	15924
$R = CD$	t/o	t/o	t/o

Timings are given in milliseconds.

Communication from Java to Mathematica

...via shell (slow but works in all versions of Mathematica)

```
static String executeMathematica_obsolete (String command, int timeLimit) {  
    String mathematicaCommand = "math";  
    if (Start.isPiUnix) {  
        mathematicaCommand = "wolfram";  
    }  
    if (Start.wolframscript) {  
        mathematicaCommand = "wolframscript";  
    }  
  
    String output;  
    int ltrim = 0;  
    command = "TimeConstrained[" + command + "," + timeLimit + "]";  
    if (Start.wolframscript) {  
        output = ExternalCAS.execute(mathematicaCommand + " -code \\""  
            + command + "\\"", timeLimit).replace("\n", "").replace("\r", "");  
        return output;  
    }  
    output = ExternalCAS.execute("echo \\""+ command + "\" | " +  
        mathematicaCommand + " | tail -n +4 | grep -v \"In\\\[2\\]\\\"", timeLimit);  
    ltrim = "In[1]:= ".length();  
    if (output.length() < ltrim) {  
        System.err.println("Error executing Mathematica command");  
        return "";  
    }  
    output = output.replace("\n>", "");  
    output = output.replace("\n", "");  
    return output.substring(ltrim);  
}
```

Communication from Java to Mathematica

...via MathLink (fast but requires full version of Mathematica)

```
static String executeMathematica (String command, int timeLimit) {
    if (Start.dryRun)
        return "";
    if (Start.wolframscript) {
        return executeMathematica_obsolete (command, timeLimit);
    }
    command = "TimeConstrained[" + command + "," + timeLimit + "]";
    String ret = ml.evaluateToInputForm(command, 0);
    // System.out.println("executeMathematica: " + command + " -> " + ret);
    return ret;
}

static boolean createMathLink () {
    String mathematicaCommand = "math";
    if (Start.isPiUnix) {
        mathematicaCommand = "wolfram";
    }
    try {
        ml = MathLinkFactory.createKernelLink("-linkmode launch -linkname '" + mathematicaCommand + " -mathlink'");
        ml.discardAnswer();
    } catch (MathLinkException e) {
        System.err.println("createMathLink: " + e.toString());
        return false;
    }
    return true;
}

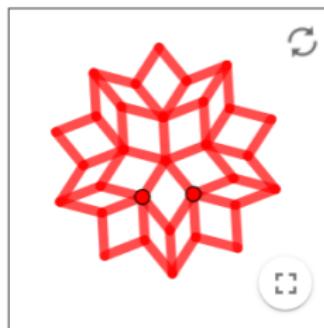
static void stopMathLink () {
    System.out.println("Stopping MathLink connection...");
    ml.close();
}
```

Thank you!

Spikey: Mathematica meets GeoGebra

Author: Zoltán Kovács

Topic: Geometry



The two red points are draggable.

See also [Stephen Wolfram's story on Spikey](#).